General Relativity $\boxed{\text{Week 2}}$

<u>Last time:</u> Lorentzian manifold (M, g)

Smooth manifold, with $p \to \{g_p : T_pM \times T_pM \to \mathbb{R}\}$ a smooth assignment of Lorentzian inner products ($g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) = g_{ij}$ is smooth in any local coordinate chart).

- A vector field $X \in \Gamma(M)$ is timelike/null/spacelike
  if $\forall p \in M$: $X_p \in T_pM$ is " / " / "
- A submanifold $S \subseteq M$ is timelike / null / spacelike
  if $\forall p \in S$: $T_pS \subseteq T_pM$ is " / " / " subspace.
- A C^1 curve $\gamma : [a,b] \to M$ is timelike / null / spacelike
  if $\dot\gamma$ is " / " / "

<u>Examples of Lorentzian manifolds</u>

- Minkowski spacetime $(\mathbb{R}^{n+1}, \eta)$: Flat spacetime

In Cartesian coordinates: $\eta = -(dx^0)^2 + (dx^1)^2 + \dots + (dx^n)^2$

So $\{\frac{\partial}{\partial x^a}\}_{a=0}^n$: Orthonormal frame at every point $p \in \mathbb{R}^{n+1}$

- A timelike curve γ: Stays inside the timecone based at every point. ①
- The straight line ~~[illegible]~~ $t \to (t, v \cdot t)$, $v \in \mathbb{R}^n$ is: timelike if $|v| < 1$, null if $|v| = 1$, spacelike if $|v| > 1$

Using ①: I can extend the definition of timelike curve to C^0 curves

Cone condition $\Rightarrow$ Lipschitz condition (so almost everywhere differentiable)

• If $(\bar{M}, \bar{g})$ is a Riemannian manifold

$$(M, g) = (\mathbb{R} \times \bar{M}, \ -dt^2 + \bar{g})$$ is Lorentzian

("ultra static" spacetime

• translations in $\mathbb{R}$ are isometries)

with $t: M \to \mathbb{R}$ being the projection on the first factor

• If (M, g) is Lorentzian and $f: M \to (0, +\infty)$

then $(M, f \cdot g)$ is also Lorentzian

($\tilde{g} = f \cdot g$ is conformal to g. In this case, $\tilde{g}$ and g have the same null cones at every point) (that's an if and only if statement)

Remarks:

• Change of coordinates: If $g = g_{\alpha\beta}\, dx^\alpha dx^\beta = g_{\alpha\beta}\, dx^\alpha \otimes dx^\beta$

and $(x^0, \dots, x^n) \longrightarrow (y^0, \dots, y^n)$

Then $g = \tilde{g}_{\alpha\beta}\, dy^\alpha dy^\beta$

But $dy^\alpha = \frac{\partial y^\alpha}{\partial x^i} dx^i$

$$g_{\mu\nu} = \tilde{g}_{\alpha\beta} \frac{\partial y^\alpha}{\partial x^\mu} \frac{\partial y^\beta}{\partial x^\nu}$$

$$\Longleftrightarrow \tilde{g}_{\alpha\beta} = g_{\mu\nu} \cdot \frac{\partial x^\mu}{\partial y^\alpha} \frac{\partial x^\nu}{\partial y^\beta}$$

In matrix notation: $[\tilde{g}] = \left[\frac{\partial x}{\partial y}\right]^T \cdot [g] \left[\frac{\partial x}{\partial y}\right]$

Recall: $\left[\frac{\partial y}{\partial x}\right] \cdot \left[\frac{\partial x}{\partial y}\right] = \mathbb{I}$ or $\frac{\partial y^\alpha}{\partial x^\mu} \frac{\partial x^\mu}{\partial y^\beta} = \delta^\alpha_\beta$

• Not every manifold admits a Lorentzian metric (unlike the Riemannian case). Topological restriction: TM must admit a line subbundle

(if M is compact: This $\Longleftrightarrow \chi(M) = 0$)

(We will prove it in the exercise)

Length of a curve: ~~[illegible]~~ $\gamma : [a,b] \to (M,g)$,

$$\ell(\gamma) = \int_a^b |\dot\gamma| \, dt$$

(makes sense for piecewise C^1 or, more generally, $W^{1,1}$ curves)

We will only consider $\ell(\gamma)$ when either γ is purely causal or purely spacelike (never for "mixed" curves)

Note: Length is independent of parametrization. If $h : [c,d] \to [a,b]$ is piecewise C^1 and increasing, then

$$\ell(\gamma \circ h) = \int_c^d \left| \frac{d}{ds}(\gamma(h(s))) \right| ds = \int_c^d |\dot\gamma(h(s))| \cdot |h'(s)| \, ds \overset{h(s)=t}{=} \int_a^b |\dot\gamma(t)| \, dt$$

Time orientation:

Let (M,g) be a Lorentzian manifold.

$\forall p \in M$: Timecone $I_p = \{ v \in T_pM : g(v,v) < 0 \}$

Has two components. Fixing a $T \in I_p$:

We can define $I_p^+ = \{ v \in I_p : g(T,v) < 0 \}$

Question: Can we choose I_p^+ for $p \in M$ in a continuous way?

- On Minkowski: Just choose $T = \frac{\partial}{\partial x^0}$.

(Similarly: Causal cone: $J_p = \{ v \in T_pM : g(v,v) \leq 0 \}$)

<u>Def</u>: A <u>time orientation</u> of (M,g) is an assignment $\forall p \in M$ of a future timecone I_p^+ which is smooth in the following sense: $\forall p \in M$, $\exists$ neighborhood U of p and smooth vector field $V \in \Gamma(U)$ such that $V_q \in I_q^+$ $\forall q \in U$.

- If (M,g) admits a time orientation: <u>Time orientable</u>
- A time oriented (M,g): <u>spacetime</u>

<u>Lemma</u>: (M,g) is time orientable <u>if and only if</u> M admits a globally timelike vector field T.

<u>Proof</u>: "$\Leftarrow$": Trivial: If T is timelike everywhere, define $\forall p \in M$: $I_p^+ = \{v \in T_pM : g(v,T) < 0\}$. Then T satisfies the definition.

"$\Rightarrow$" Let I_p^+ be the chosen assignment. Let $\{U_a\}_{a \in A}$ be a collection of open sets as in the definition covering M, i.e.:

- $\bigcup_{a \in A} U_a = M$ and
- $\forall a \in A$, $\exists$ $T_a \in \Gamma(U_a)$ with $T_a|_q \in I_q^+$ $\forall q \in U_a$.

Let $\{\chi_\beta\}_{\beta \in B}$ be a <u>subordinate partition of unity</u> (so $\forall \beta$, $\exists a(\beta)$ s.t. $\operatorname{supp} \chi_\beta \subseteq U_{a(\beta)}$).

Then define $T = \sum_\beta \chi_\beta \cdot T_{a(\beta)}$.

The sum is well defined: $T_{a(\beta)}$ is defined on $\operatorname{supp} \chi_\beta$.

and the sum is finite around every point.

So: T is a smooth vector field.

We will show it is everywhere timelike:

$\forall p \in M$:, $\forall \beta \in B$: $\chi_\beta \cdot T_{a(\beta)}\big|_p \in \underbrace{I_p^+ \cup \{0\}}_{\text{convex cone!}}$

and at least one of them non-zero

$$\Longrightarrow \sum_\beta \chi_\beta \cdot T_{a(\beta)}\Big|_p \in I_p^+. \qquad \blacksquare$$

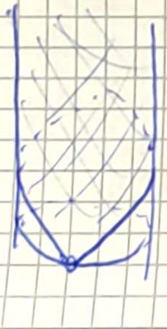
Definition: Let (M,g) be time oriented.

- Timelike future: $I^+[p] \subseteq M$, defined as

$$I^+[p] = \left\{ q \in M : \exists \gamma : [0,1] \to M, \ \gamma(0) = p, \ \gamma(1) = q, \ \dot{\gamma}(t) \in I^+_{\gamma(t)} \ \forall t \in [0,1] \right\}$$

- Causal future: $J^+[p] = \left\{ q \in M : \text{"} \quad \text{"} \quad \dot{\gamma}(t) \in J^+_{\gamma(t)} \ \forall t \in [0,1] \right\}$

Examples:
- On Minkowski spacetime: $I^+[0] = I^+_0$ if we identify $\mathbb{R}^{n+1}$ with $T_0\mathbb{R}^{n+1}$
- On $\mathbb{R} \times S^1$ with $g = -dt^2 + dx^2$ (that's a quotient of Minkowski):



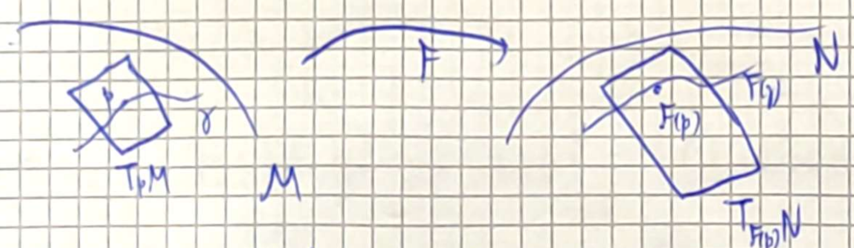
<u>Note</u>: • Time orientability $\neq$ topological orientability

• In general: $I^+[p]$ can contain a whole neighborhood of p.

(check $\mathbb{S}^1 \times \mathbb{S}^1$, $g = -dt^2 + dx^2$)

But: Restricting g to a small neighborhood U of p:
Similar picture as for Minkowski.

Some more stuff from the theory of manifolds:



Let $F: M \to N$ be smooth

Differential of F: $\quad dF_p : T_pM \longrightarrow T_{F(p)}N \quad$ (linear)

If γ is a curve: $\quad \dot{(F\circ\gamma)} = dF(\dot{\gamma})$

In local coordinates: $\quad \left(dF\left(\frac{\partial}{\partial x^i}\right)\right)^a = \frac{\partial F^a}{\partial x^i}$

So $\quad (dF(X))^a = \frac{\partial F^a}{\partial x^i} X^i$

<u>Definition</u>: 1) F is an <u>immersion</u> if dF is <u>injective</u> everywhere

$(dF_p : T_pM \to T_pN)$

2) F is an <u>embedding</u> if immersion & 1-1 & homeomorphism on its image

3) <u>Diffeomorphism</u> if bijective and both F & F^{-1} are embeddings

Pull-back of metric:

Def: Let $F: M \to N$ be an immersion, h: a symmetric $(0,2)$ tensor field on N.

$$g = F^* h \quad : \quad g(X,Y) = h(dF(X), dF(Y)).$$

In local coordinates: $g_{\alpha\beta} = h_{\mu\nu} \frac{\partial F^\mu}{\partial x^\alpha} \frac{\partial F^\nu}{\partial x^\beta}$

- If h is Riemannian: F^*h is also Riemannian
- If h is Lorentzian: F^*h could be anything (even degenerate)

Definition: $F: (M,g) \to (N,h)$ (Lorentzian manifolds) is

- A local isometry if $dF_p : (T_pM, g_p) \to (T_pN, h_p)$ is a linear isometry $\forall p$
- A global isometry if it is a local isometry & bijective
- A local conformal isometry if $dF_p : (T_pM, g_p) \to (T_pN, h)$ is a similarity transformation

Note:
All global isometries from $(\mathbb{R}^{n+1}, \eta)$ to itself are affine maps, belonging to $O(1,n) \ltimes \mathbb{R}^{1+n}$
(see Exercise)

Tensors and tensor products:

Let M be a smooth manifold and $p \in M$.

Def: The space of (k,l) tensors at p is the set of multilinear maps

$$T : \underbrace{T_p^*M \times \dots \times T_p^*M}_{k} \times \underbrace{T_pM \times \dots \times T_pM}_{l} \longrightarrow \mathbb{R}.$$

- Tangent vector: Type $(1,0)$ $\quad (v : T_p^*M \to \mathbb{R})$
- Cotangent vector: $(0,1)$
- Lorentzian metric: $(0,2)$

Tensor product: If $\left.\begin{matrix} \omega_1 \text{ of type } (k_1, l_1) \\ \omega_2 \text{ of type } (k_2, l_2) \end{matrix}\right\} \omega_1 \otimes \omega_2 : (k_1+k_2, l_1+l_2)$,

$$\omega_1 \otimes \omega_2 (y_1, \dots, y_{k_1+k_2}; v_1, \dots, v_{l_1+l_2}) = \omega_1(y_1, \dots, y_{k_1}; v_1, \dots, v_{l_1}) \cdot \omega_2(y_{k_1+1}, \dots, y_{k_1+k_2}; v_{l_1+1}, \dots, v_{l_1+l_2}).$$

Not commutative

In local coordinates: Basis for tensors of type (k,l):

$$\left\{ \frac{\partial}{\partial x^{i_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{i_k}} \otimes dx^{j_1} \otimes \dots \otimes dx^{j_l} \right\}_{\substack{i_1, \dots, i_k \\ j_1, \dots, j_l = 0}}^{n}.$$

In this basis: Components of tensors:

$$T = T^{\overbrace{i_1 \dots i_k}^{\text{contravariant indices}}}_{\underbrace{j_1 \dots j_l}_{\text{covariant indices}}} \frac{\partial}{\partial x^{i_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{i_k}} \otimes dx^{j_1} \otimes \dots \otimes dx^{j_l},$$

where $T^{i_1 \dots i_k}_{j_1 \dots j_l} = T\left(dx^{i_1}, \dots, dx^{i_k}; \frac{\partial}{\partial x^{j_1}}, \dots, \frac{\partial}{\partial x^{j_l}}\right)$

Change of coordinates:

$$(x^0, \dots, x^n) \longrightarrow (y^0, \dots, y^n),$$

$$\tilde{T}^{i_1 \dots i_k}_{j_1 \dots j_l} = T^{\alpha_1 \dots \alpha_k}_{\beta_1 \dots \beta_l} \frac{\partial y^{i_1}}{\partial x^{\alpha_1}} \cdots \frac{\partial y^{i_k}}{\partial x^{\alpha_k}} \cdot \frac{\partial x^{\beta_1}}{\partial y^{j_1}} \cdots \frac{\partial x^{\beta_l}}{\partial y^{j_l}} .$$